BUSINESS CYCLE TURNING POINTS, A NEW COINCIDENT INDEX, AND TESTS OF DURATION DEPENDENCE BASED ON A DYNAMIC FACTOR MODEL WITH REGIME SWITCHING

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Abstract—The synthesis of the dynamic factor model of Stock and Watson (1989) and the regime-switching model of Hamilton (1989) proposed by Diebold and Rudebusch (1996) potentially encompasses both features of the business cycle identified by Burns and Mitchell (1946): (1) comovement among economic variables through the cycle and (2) nonlinearity in its evolution. However, maximum-likelihood estimation has required approximation. Recent advances in multimove Gibbs sampling methodology open the way to approximation-free inference in such non-Gaussian, nonlinear models. This paper estimates the model for U.S. data and attempts to address three questions: Are both features of the business cycle empirically relevant? Might the implied new index of coincident indicators be a useful one in practice? Do the resulting estimates of regime switches show evidence of duration dependence? The answers to all three would appear to be yes.

I. Introduction

IN THEIR pioneering study, Burns and Mitchell (1946) established two defining characteristics of the business cycle: (1) comovement among economic variables through the cycle and (2) nonlinearity in the evolution of the business cycle, that is, regime switching at the turning points of the business cycle. As noted by Diebold and Rudebusch (1996), these two aspects of the business cycle have generally been considered in isolation from one another in the literature. Two of the most recent and influential examples include Stock and Watson's (1989, 1991, 1993) linear dynamic factor model, in which comovement among economic variables is captured by a composite index, and Hamilton's (1989) regime-switching model, featuring nonlinearity in an individual economic variable. After an extensive survey of related literature, and based on both theoretical and empirical foundations, Diebold and Rudebusch (1996) propose a multivariate dynamic factor model with regime switching in which the two key features of the business cycle are encompassed.

The purpose of this paper is twofold. First, we develop approximation-free inference in the Diebold and Rudebusch dynamic factor model with regime switching. As in Stock and Watson (1991), the model can be cast in state–space form, but nonlinearity (due to regime switching) in the transition equation implies that the usual Gaussian Kalman filter cannot be applied directly.[1] Recent advances in Gibbs sampling methods, however, make approximation-free inference[2] for non-Gaussian, nonlinear state–space models feasible. In particular, we take advantage of the multimove Gibbs sampling methodology of Shephard (1994) and Carter and Kohn (1994) and estimate the model in a Bayesian framework in which the parameters, the regime that the economy is in, and the dynamic factor or composite index are all treated as unobserved random variables to be inferred from data on individual indicators. Using this approach, we reevaluate the empirical relevance of the two key features of the business cycle identified by Burns and Mitchell (1946). As by-products, we are able to calculate a new experimental coincident index and the regime probabilities at each point in time.

The second objective of this paper is to test for business cycle duration dependence, whether the probability of a transition between regimes depends on how long the economy has been in a recession or boom. While most of existing literature on tests of business cycle duration dependence deals with the issue within univariate contexts, we offer ways to deal with such issues within a multivariate and Bayesian context by incorporating nonzero probabilities of duration dependence in the Diebold and Rudebusch dynamic factor model with regime switching.

The structure of this paper is as follows. Section II introduces the model and discusses related econometric issues. In section III the multimove Gibbs sampling of Shephard (1994) and Carter and Kohn (1994) is applied to our model in order to obtain Bayesian estimates of the unobserved regime-indicator variable and the unobserved common growth component. The implied new composite index of coincident indicators is presented and the empirical relevance of regime switching and comovement among economic variables is evaluated. Section IV presents tests for business cycle duration dependence that exploit the Gibbs sampling methodology. The basic model is extended to incorporate nonzero probabilities of duration dependence. Section V concludes the paper.

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[1] For a general approach to dealing with nonlinear and non-Gaussian state–space models, readers are referred to Kitagawa (1987). Various approximate filtering and smoothing algorithms for nonlinear and non-Gaussian state models are given in Harrison and Stevens (1976), Gordon and Smith (1988), Shumway and Stoffer (1991), and Kim (1993a, b, 1994). Kim and Yoo (1995) and Chauvet (1995) apply Kim's (1993a, b, 1994) approximate maximum-likelihood estimation method in estimating a dynamic factor model with regime switching.

[2] A referee has suggested that it is not entirely accurate to say that the Gibbs sampling framework makes no approximations. The key approximation in the Gibbs framework is associated with declaring the Gibbs chain to have converged to its steady state. It is in the limit, as the length of the chain goes to infinity, that the approximation error vanishes.

II. A Dynamic Factor Model of the Business Cycle with Regime Switching

In the synthesis of the dynamic factor model of Stock and Watson (1989, 1991, 1993) and the regime-switching model of Hamilton (1989) proposed by Diebold and Rudebusch (1996), the growth rate of each of four observed individual coincident indicators depends on current and lagged values of an unobserved common factor, which is interpreted as the composite index of coincident indicators. The growth rate of the index is, in turn, dependent upon whether the economy is in the recession state or in the boom state. The model may be written as

$$\Delta Y_{it} = \lambda_i(L)\Delta C_t + D_i + e_{it}, \qquad i = 1, 2, 3, 4, \quad t = 1, 2, \ldots, T \tag{1}$$

where ΔY_{it} represents the first difference of the log of the ith indicator, $i = 1, \ldots, 4$; $\lambda_i(L)$ is the polynomial in the lag operator; ΔC_t is the growth rate of the composite index; D_i is an intercept for the ith indicator; and e_{it} is a process with autoregressive (AR) representation,

$$\psi_i(L)e_{it} = \epsilon_{it}, \qquad \epsilon_{it} \sim \text{iid } N(0, \sigma_i^2). \tag{2}$$

Thus each indicator ΔY_{it}, $i = 1, 2, 3, 4$, consists of an individual component $(D_i + e_{it})$ and a linear combination of current and lagged values of the common factor or index ΔC_t. The index is assumed to be generated by the AR process,

$$\phi(L)(\Delta C_t - \mu_{s_t} - \delta) = v_t, \qquad v_t \sim \text{iid } N(0, 1) \tag{3}$$

and v_t and ϵ_{it} are independent of one another for all t and i, while the variance of v_t is taken to be unity for identification of the model. While δ is constant over time, μ_{s_t} depends on whether the economy is in a recession ($S_t = 0$) or in a boom ($S_t = 1$), as follows:

$$\mu_{s_t} = \mu_0 + \mu_1 S_t, \qquad \mu_1 > 0, \quad S_t = \{0, 1\}. \tag{4}$$

Transitions between regimes or states of the economy are then governed by the Markov process,

$$\Pr[S_t = 1 | S_{t-1} = 1] = p, \quad \Pr[S_t = 0 | S_{t-1} = 0] = q. \tag{5}$$

This model differs from Stock and Watson's (1991) linear dynamic factor model of the business cycle by allowing the mean growth rate of the coincident index, given by $(\mu_{s_t} + \delta)$, to switch between the two regimes of the business cycle. We note that the means of the variables are overdetermined in this parameterization. Imposing a mean of zero on the μ_{s_t} process, δ determines the long-run growth rate of the index whereas μ_{s_t} produces deviations from that long-run growth rate according to whether the economy is in a recession or in a boom.

This model can be cast into state–space form, but there is not a unique state–space representation. Depending on our purpose, we may choose a particular state–space representation, and in this section we focus on the following one:

$$\Delta Y_t = H\xi_t + \tilde{D} \tag{6}$$

$$\xi_t = \tilde{M}_{s_t} + \tilde{\delta} + F\xi_{t-1} + u_t \tag{7}$$

where $\Delta Y_t = [\Delta Y_{1t} \;\; \Delta Y_{2t} \;\; \Delta Y_{3t} \;\; \Delta Y_{4t}]'$, $\tilde{D} = [D_1 \;\; D_2 \;\; D_3 \;\; D_4]'$, and the other tems are defined appropriately, according to the specifications of $\phi(L)$, $\psi_i(L)$, and $\lambda_i(L)$. Assuming, for example, an AR(1) common component and AR(1) individual components, and $\lambda_i(L) = \lambda_i$, $i = 1, 2, 3, 4$, we would have

$$H = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ \lambda_2 & 0 & 1 & 0 & 0 \\ \lambda_3 & 0 & 0 & 1 & 0 \\ \lambda_4 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$F = \begin{bmatrix} \phi_1 & 0 & 0 & 0 & 0 \\ 0 & \psi_{11} & 0 & 0 & 0 \\ 0 & 0 & \psi_{21} & 0 & 0 \\ 0 & 0 & 0 & \psi_{31} & 0 \\ 0 & 0 & 0 & 0 & \psi_{41} \end{bmatrix}, \qquad \xi_t = \begin{bmatrix} \Delta C_t \\ e_{1t} \\ e_{2t} \\ e_{3t} \\ e_{4t} \end{bmatrix},$$

$$u_t = \begin{bmatrix} v_t \\ \epsilon_{1t} \\ \epsilon_{2t} \\ \epsilon_{3t} \\ \epsilon_{4t} \end{bmatrix}, \qquad \tilde{M}_{s_t} = \begin{bmatrix} (1 - \phi_1 L)\mu_{s_t} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \tilde{\delta} = \begin{bmatrix} (1 - 0)\delta \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since the parameters $\tilde{D}$ and $\tilde{\delta}$ overdetermine the means of the processes, as alluded to above, the model is not identified. However, when the data are expressed as deviations from means ($\Delta y_{it} = \Delta Y_{it} - \Delta \bar{Y}_i$), equations (1)–(3) or

(6)–(7) can be replaced by

$$\Delta y_{it} = \lambda_i(L)\Delta c_t + e_{it} \quad (8)$$

$$\phi(L)(\Delta c_t - \mu_{s_t}) = v_t \quad (9)$$

where $\Delta c_t = \Delta C_t - \delta$, or in state–space form,

$$\Delta y_t = H\xi_t^* \quad (10)$$

$$\xi_t^* = \tilde{M}_{s_t} + F\xi_{t-1}^* + u_t \quad (11)$$

where the first element of ξ_t^* is given by Δc_t instead of ΔC_t. Thus in computing the log likelihood based on the Kalman filter, the terms contained in $\tilde{D}$ and $\tilde{\delta}$ can be concentrated out of the likelihood function.

If the business cycle regime, S_t, were observable, then this would be a linear Gaussian model and the procedures described in Stock and Watson (1991) based on the likelihood function could be applied to: (1) estimate the parameters of the model based on the state–space representation in equations (10) and (11); (2) recover D_i and δ from $\Delta \overline{Y} = [\Delta \overline{Y}_1 \ \cdots \ \Delta \overline{Y}_4]'$ based on the steady-state Kalman gain; and (3) calculate the composite coincident index C_t. However, since the regime is not directly observed, rather it must be inferred from the data, the usual Kalman filter cannot be employed to carry out Stock and Watson's (1991) procedures. An intuitive explanation for this is as follows. Each iteration of the Kalman filter produces a twofold increase in the number of cases to consider, as explained in Harrison and Stevens (1976), Gordon and Smith (1988), and Kim (1993a, b, 1994). Since S_t takes on two possible values in each time period, there would be 2^T possible paths to consider in evaluating the conditional log likelihood. For the monthly data used in this paper, T is about 400, implying an impractical computational burden. Thus to make Stock and Watson's (1991) procedures operable, approximations to the Kalman filter are unavoidable.

Kim and Yoo (1995) and Chauvet (1995) employ the approximate maximum-likelihood estimation method of Kim (1993a, b, 1994) to estimate the nonlinear dynamic factor model of the business cycle.[3] Though the maximum-likelihood estimation based on approximations to the Kalman filter is straightforward to implement, it is difficult to judge the effects of the approximations on the parameter estimates and on inferences pertaining to the unobserved common component ξ_t^* and the unobserved state S_t. The effects of the approximation on the steady-state Kalman gain are also unknown, and this uncertainty complicates the task of decomposing $\Delta \overline{Y}_i$ into D_i and δ, which is necessary to extract the composite coincident index $C_t = \Delta c_t + C_{t-1} + \delta$. In addition, since inferences about S_t and ξ_t^*, and thus Δc_t, are based on final parameter estimates of the model, effects of parameter uncertainty are also unknown in this approach.

In contrast, the Gibbs sampling approach described in the next section incorporates parameter uncertainty by casting the model in a Bayesian framework, and no approximations are used in estimating parameters or in making inferences about S_t, ξ_t^*, and thus Δc_t. In addition, it provides us with more information than may be extracted from an approximate maximum-likelihood estimation of the model. For example, posterior distributions of the parameters and other variates contain important information, which we exploit in section IV.

III. Bayesian Inference in the Dynamic Factor Model with Regime Switching Using Gibbs Sampling—Fixed Transition Probabilities

The objective is to infer from data on the four coincident indicators: (1) the path of the growth rate in the composite index Δc_t, $t = 1, 2, \ldots, T$, represented by $\Delta \tilde{c}_T$, (2) the path of the Markov switching regime indicator S_t, $t = 1, 2, \ldots, T$, represented by $\tilde{S}_T$, and (3) all unknown parameters of the model, represented by a K-dimensional vector $\tilde{\theta}$. In the Bayesian framework these are viewed as three vectors of random variables, and inference amounts to summarizing their joint distribution, conditional on historical data on the four individual coincident indicator variables. For reasons discussed above, this joint distribution is not readily obtained directly, but the fact that the distribution of any one of these vectors, conditional on the other two and the data, can be obtained opens the problem up to solution by Gibbs sampling. In brief, Gibbs sampling[4] is an iterative Monte Carlo technique that generates a simulated sample from the joint distribution of a set of random variables by generating successive samples from their conditional distributions. In the present case, Gibbs sampling proceeds by taking a drawing from the conditional distribution of $\Delta \tilde{c}_T$ given the data $\tilde{S}_T$ and $\tilde{\theta}$; then a drawing from the conditional distribution of $\tilde{S}_T$ given the data, the prior drawing of $\Delta \tilde{c}_T$, and $\tilde{\theta}$; and then a drawing from the conditional distribution of $\tilde{\theta}$ given the data and prior drawings of $\Delta \tilde{c}_T$ and $\tilde{S}_T$. By successive iteration, the procedure simulates (perhaps surprisingly) a drawing from the joint distribution of the three vectors consisting of the $2T + K$ variates of interest, given the data. It is straightforward then to summarize the marginal distributions of any of these, given the data. Recently Carter and Kohn (1994) and Shephard (1994) introduced an efficient multimove Gibbs sampling approach for state–space models, with which we can generate all Δc_t at once by taking advantage of the time ordering of the state–space model. They also show that their multimove Gibbs sampling dominates Carlin et al.'s (1992) single-move Gibbs sampling, especially in terms of efficiency and the convergence to the posterior distribution.

[3] Kim and Yoo (1995) and Chauvet (1995) estimate modified versions of the model given in equations (8) and (9). Instead of allowing the mean of the common component to be regime dependent, they allow an intercept term to be regime dependent.

[4] For a general introduction to Gibbs sampling, readers are referred to Gelfand and Smith (1990).

More specifically, the three steps involved in generating the Gibbs sample in our model are as follows. First, conditional on $\tilde{S}_T = [S_1 \quad S_2 \quad \cdots \quad S_T]'$ and $\tilde{\theta}$ (all unknown parameters of the model), the model is a linear Gaussian state–space model from which we can generate $\Delta\tilde{c}_T = [\Delta c_1 \quad \Delta c_2 \quad \cdots \quad \Delta c_T]'$ using the multimove Gibbs sampling technique. Second, conditional on $\Delta\tilde{c}_T$ and $\tilde{\theta}$, we can then focus on equation (9) for the common component to generate $\tilde{S}_T$. This is possible because the common component is assumed to be distributed independently of individual components. The procedure provided by Albert and Chib (1993) could be used, but as in the case of $\Delta\tilde{c}_T$, the multimove Gibbs sampling can be implemented instead. Third and finally, conditional on $\Delta\tilde{c}_T$ and $\tilde{S}_T$, we are left with generating all unknown parameters of the model $\tilde{\theta}$. As the four equations in the model are independent of one another except for the common component, we can treat them separately. Given this structure, the problem of generating the parameters of each equation essentially collapses to a form of Bayes' regression with autocorrelated errors, as proposed by Chib (1993). Further details of the procedures used for generating $\Delta\tilde{c}_T$, $\tilde{S}_T$, and $\tilde{\theta}$ and, finally, calculation of the new composite index of coincident indicators in levels C_t, $t = 1, 2, \ldots, T$, are described in appendix A.

The four monthly series for the United States used in the estimation phase of this study are those used by the Department of Commerce (DOC) to construct its composite index of coincident indicators: industrial production (*IP*), total personal income less transfer payments in 1987 dollars (*GMYXPQ*), total manufacturing and trade sales in 1987 dollars (*MTQ*), and employees on nonagricultural payrolls (*LPNAG*).[5] The time period is January 1960 through January 1995.

Second-order autoregressive specifications are adopted for the error processes of both the common component and the four idiosyncratic components in equations (2) and (3). Stock and Watson (1989, 1991) point out that the payroll variable *LPNAG* may not be exactly coincident, lagging slightly the unobserved common component. Following them, three lags of Δc_t are included in the *LPNAG* equation, the fourth equation in equations (8). Readers are referred to appendix B for the two alternative state–space representations of the model actually employed in this paper. Different initial values of the parameters resulted in different speeds of convergence for the Gibbs sampler. To be on the safe side, the first 2000 draws in the Gibbs simulation process are discarded, and then the next 8000 draws are saved and used to calculate moments of the posterior distribution. This procedure ensures that the results are not simply an artifact of the initial values.[6] The 95% posterior probability bands are based on the 2.5th and the 97.5th percentiles of the 8000 simulated draws.

TABLE 1.—BAYESIAN PRIOR AND POSTERIOR DISTRIBUTIONS INFORMATIVE PRIORS ON FIXED TRANSITION PROBABILITIES

	Prior		Posterior			
	Mean	SD	Mean	SD	MD	95% Bands
ΔC_t						
q	0.9	0.066	0.875	0.044	0.881	(0.775, 0.945)
p	0.967	0.032	0.976	0.009	0.977	(0.954, 0.991)
ϕ_1	0.0	1	0.313	0.076	0.313	(0.165, 0.471)
ϕ_2	0.0	1	−0.002	0.067	−0.002	(−0.134, 0.131)
μ_0	0.0	1	−1.777	0.294	−1.779	(−2.346, −1.193)
μ_1	0.0	1	2.110	0.302	2.117	(1.500, 2.686)
δ	—	—	0.568	0.038	0.565	(0.500, 0.650)
$\mu_0 + \mu_1$	—	—	0.333	0.104	0.336	(0.114, 0.526)
Δy_{1t}						
λ_1	0.0	1	0.568	0.036	0.568	(0.497, 0.641)
ψ_{11}	0.0	1	−0.003	0.066	−0.005	(−0.132, 0.125)
ψ_{12}	0.0	1	−0.030	0.064	−0.030	(−0.154, 0.096)
σ_1^2	—	—	0.238	0.033	0.237	(0.175, 0.309)
Δy_{2t}						
λ_2	0.0	1	0.213	0.021	0.212	(0.173, 0.254)
ψ_{21}	0.0	1	−0.302	0.050	−0.302	(−0.404, −0.204)
ψ_{22}	0.0	1	−0.070	0.050	−0.069	(−0.167, 0.031)
σ_2^2	—	—	0.317	0.024	0.316	(0.273, 0.366)
Δy_{3t}						
λ_3	0.0	1	0.442	0.034	0.441	(0.377, 0.512)
ψ_{31}	0.0	1	−0.354	0.053	−0.353	(−0.458, −0.244)
ψ_{32}	0.0	1	−0.154	0.052	−0.154	(−0.258, −0.048)
σ_3^2	—	—	0.660	0.051	0.658	(0.565, 0.767)
Δy_{4t}						
λ_{40}	0.0	1	0.120	0.010	0.120	(0.102, 0.140)
λ_{41}	0.0	1	0.006	0.009	0.006	(−0.011, 0.024)
λ_{42}	0.0	1	0.023	0.009	0.023	(0.006, 0.040)
λ_{43}	0.0	1	0.026	0.008	0.026	(0.012, 0.041)
ψ_{41}	0.0	1	−0.022	0.060	−0.021	(−0.141, 0.092)
ψ_{42}	0.0	1	0.277	0.061	0.280	(0.155, 0.390)
σ_4^2	—	—	0.021	0.002	0.021	(0.017, 0.025)

Notes: (1) y_{1t}, y_{2t}, y_{3t}, and y_{4t} represent *IP*, *GMYXPQ*, *MTQ*, and *LPNAG*, respectively.
(2) Prior distribution of σ_i^2 is improper.
(3) SD and MD refer to standard deviation and median, respectively.
(4) 95% bands refers to 95% posterior probability bands.

Table 1 presents the Bayesian prior and posterior distributions of the parameters. For each parameter, the prior distribution is specified by mean and standard deviation (SD), and the posterior by mean, standard deviation (SD), median (MD), and the 95% posterior probability bands. Rather tight prior distributions for p and q are designed to incorporate what was known or believed about the average duration of business cycle phases by the beginning of the sample period. The average duration of recessions and booms implied by the means of these priors are 10 months and 33.3 months, respectively. As the model is estimated with data expressed in deviation from means, under the hypothesis of no regime switching we have $\mu_0 = 0$ and $\mu_0 + \mu_1 = 0$. Standard errors and the 95% posterior probability bands for these parameters seem to provide evidence in favor of Burns and Mitchell's (1946) concept of the division of the business cycle into two separate phases.

[5] The abbreviations, *IP*, *GMYXPQ*, *MTQ*, and *LPNAG* are DRI variable names.

[6] The initial values employed are $p = q = 0.9$; $\phi_1 = \phi_2 = 0$; $\lambda_i = 0.5$, $i = 1, 2, 3, 4$; $\lambda_{4j} = 0, j = 1, 2, 3$; $\psi_{ij} = 0, i = 1, 2, 3, 4, j = 1, 2$; $\sigma_i^2 = 0.2, i = 1, 2, 3, 4$. Gelman and Rubin (1992) suggest that a single sequence of samples may give a false impression of convergence, no matter how many draws are chosen. Thus following one of the referees, we also tried various different initial values. The posterior distributions were robust with respect to different initial values.

FIGURE 1.—PROBABILITIES OF A RECESSION (MULTIVARIATE MODEL; INFORMATIVE PRIORS)

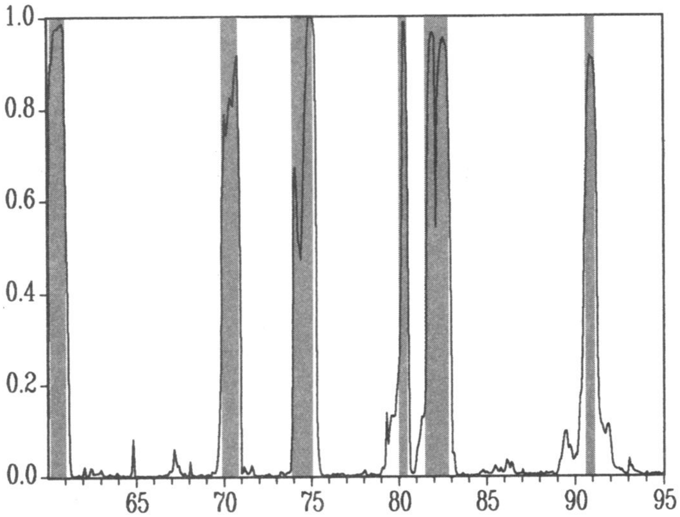


FIGURE 2.—PROBABILITIES OF A RECESSION (UNIVARIATE MODEL; INFORMATIVE PRIORS)

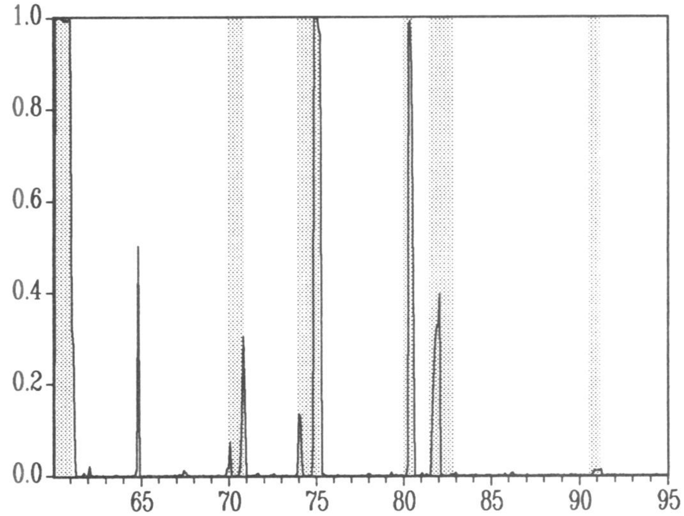


Figure 1 depicts the posterior probability that the economy was in the recession regime in each month as inferred from the Gibbs sampling. The shaded areas represent the periods of National Bureau of Economic Research (NBER) recessions (from peak to trough). The posterior probabilities are in close agreement with the NBER reference cycle. To what extent do these results reflect an ability of the model to extract Burns and Mitchell's (1946) "comovement of economic variables" over the business cycle, or do these results depend importantly on our use of tight priors for transition probabilities?

In order to evaluate the marginal contribution of going from a univariate analysis to a multivariate common factor approach, we reestimated the posterior regime probabilities from a univariate regime-switching model of the IP series, one of the four series employed in this paper. The same tight priors were given for the transition probabilities of the univariate model, as in the multivariate case.[7] Figure 2 shows the results. A univariate analysis results in significantly lower correlation between posterior regime probabilities and the NBER reference cycle.[8]

In order to evaluate the marginal contribution of the use of tight priors for p and q, the multivariate model was reestimated using noninformative priors on these parameters. Priors were chosen such that expected durations for booms and recessions are both two months (i.e., expected values of the priors on p and q were both specified as 0.5). Except that the 95% posterior probability bands are somewhat larger than in the case of tight priors, results reported in table 2 are almost the same as in the case of tight priors. In figure 3 posterior recession probabilities from the noninformative priors are depicted. One can hardly distinguish figure 3 from figure 1. Thus prior information about transition probabilities does not play an important role in infering

[7] In a univariate context, Filardo and Gordon (1993) take a similar approach in evaluating the marginal contributions of the tight priors for transition probabilities versus the use of leading indicator data in a model of time-varying transition probabilities.

[8] Filardo (1994) and Kim (1996) also report similar findings for individual monthly data, based on maximum-likelihood estimation of the univariate regime-switching model.

TABLE 2.—BAYESIAN PRIOR AND POSTERIOR DISTRIBUTIONS NONINFORMATIVE PRIORS ON FIXED TRANSITION PROBABILITIES

	Prior		Posterior			
	Mean	SD	Mean	SD	MD	95% Bands
ΔC_t						
q	0.5	0.289	0.808	0.133	0.841	(0.325, 0.938)
p	0.5	0.289	0.946	0.108	0.972	(0.568, 0.989)
ϕ_1	0.0	1	0.341	0.102	0.328	(0.170, 0.575)
ϕ_2	0.0	1	0.001	0.069	0.002	(−0.129, 0.138)
μ_0	0.0	1	−1.659	0.563	−1.779	(−2.435, −0.044)
μ_1	0.0	1	1.968	0.612	2.097	(0.105, 2.740)
δ	—	—	0.569	0.039	0.566	(0.501, 0.650)
$\mu_0 + \mu_1$	—	—	0.308	0.192	0.316	(−0.029, 0.529)
Δy_{1t}						
λ_1	0.0	1	0.571	0.040	0.569	(0.498, 0.654)
ψ_{11}	0.0	1	−0.012	0.066	−0.011	(−0.139, 0.121)
ψ_{12}	0.0	1	−0.033	0.064	−0.033	(−0.158, 0.094)
σ_1^2	—	—	0.237	0.033	0.236	(0.175, 0.305)
Δy_{2t}						
λ_2	0.0	1	0.215	0.022	0.214	(0.174, 0.259)
ψ_{21}	0.0	1	−0.303	0.052	−0.302	(−0.405, −0.206)
ψ_{22}	0.0	1	−0.069	0.051	−0.070	(−0.168, 0.031)
σ_2^2	—	—	0.317	0.024	0.316	(0.275, 0.365)
Δy_{3t}						
λ_3	0.0	1	0.445	0.037	0.445	(0.376, 0.522)
ψ_{31}	0.0	1	−0.354	0.053	−0.354	(−0.458, −0.249)
ψ_{32}	0.0	1	−0.153	0.053	−0.153	(−0.257, −0.049)
σ_3^2	—	—	0.660	0.052	0.659	(0.567, 0.768)
Δy_{4t}						
λ_{40}	0.0	1	0.121	0.010	0.121	(0.102, 0.143)
λ_{41}	0.0	1	0.006	0.009	0.006	(−0.012, 0.024)
λ_{42}	0.0	1	0.023	0.009	0.024	(0.006, 0.040)
λ_{43}	0.0	1	0.026	0.008	0.026	(0.011, 0.042)
ψ_{41}	0.0	1	−0.020	0.058	−0.019	(−0.136, 0.096)
ψ_{42}	0.0	1	0.282	0.060	0.283	(0.162, 0.394)
σ_4^2	—	—	0.021	0.002	0.021	(0.017, 0.025)

Notes: (1) y_{1t}, y_{2t}, y_{3t}, and y_{4t} represent *IP, GMYXPQ, MTQ,* and *LPNAG*, respectively.
(2) Prior distribution of σ_i^2 is improper.
(3) SD and MD refer to standard deviation and median, respectively.
(4) 95% bands refers to 95% posterior probability bands.

FIGURE 3.—PROBABILITIES OF A RECESSION (MULTIVARIATE MODEL; NONINFORMATIVE PRIORS)

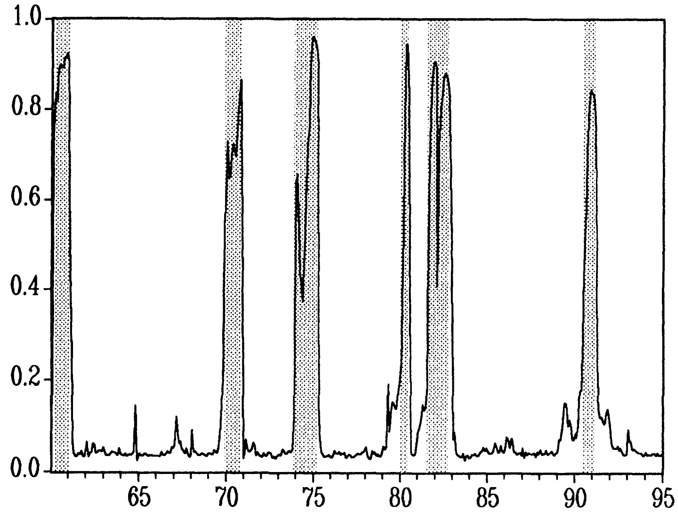
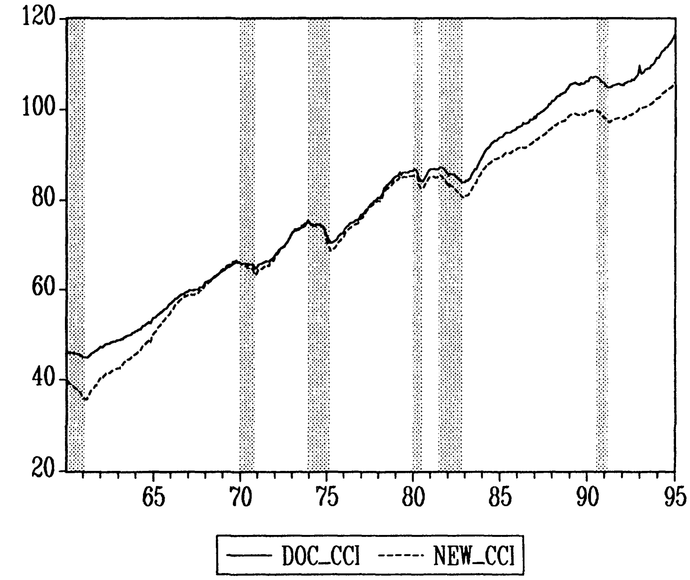


FIGURE 4.—NEW COINCIDENT INDEX VERSUS DOC'S COINCIDENT INDEX



whether the economy is in the recession or the boom regime. This, along with the substantially weaker results from the univariate model, suggests that it is the ability to capture "comovement among economic variables" in a coincident index that accounts for the model's success in identifying the NBER turning points.

In figure 4 the composite coincident index implied by the model is plotted together with the DOC's coincident index.[9] Contemporaneous correlation between the first differences of the two series is 0.9825. Though the model-based index agrees closely with the DOC index, during recessions the decline in the growth rate of the model-based index is much sharper than that of the DOC index. In the early portion of the sample, the growth rate of the model-based index during booms seems to be higher than that of the DOC index, but in the more recent portion of the sample, the pattern in the growth rates during booms seems to be reversed. After the 1982 recession, the model-based index seems to show slower growth than the DOC coincident index.

It might also be useful to compare the new coincident index with the one based on Stock and Watson's (1991) linear dynamic factor model. Table 3 presents the Bayesian prior and posterior distributions of the parameters for the Stock and Watson model without the feature of regime switching. The posterior distributions of parameters are quite close to those from the model with regime switching. One exception is that the sum of the AR coefficients ($\phi_1 + \phi_2$) for the common component is higher for the Stock and Watson model. In figure 5 the composite coincident indexes implied by the two models with and without regime switching are plotted and compared. The two indexes are almost identical, except that since the 1970s, the Stock and Watson index seems to show higher growth than the new experimental index from a dynamic factor model with

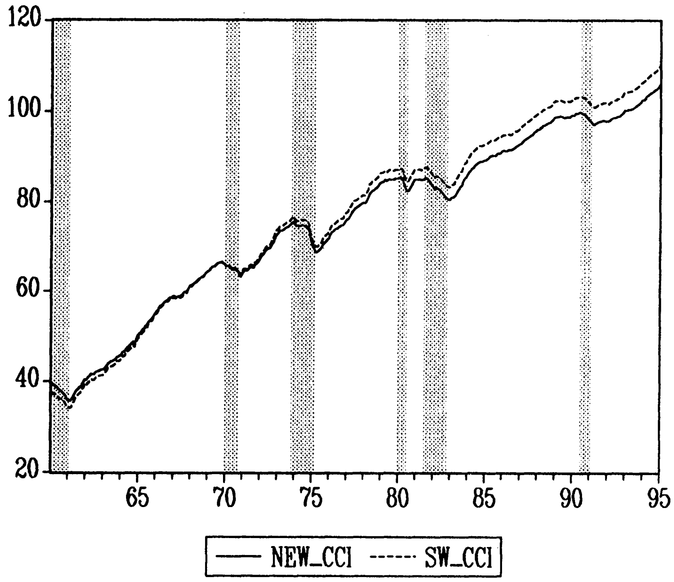
[9] The model-based index in figure 4 is calculated first by scaling ΔC_t from the model to have the same variance as the growth in the DOC coincident index. The scaled ΔC_t is maintained to have the same mean as the original ΔC_t. Then the new index is adjusted such that it is equal to the value of the DOC index in January 1970. The coincident index derived with noninformative priors was almost identical to the one with tight priors, and thus is not shown here.

TABLE 3.—BAYESIAN PRIOR AND POSTERIOR DISTRIBUTIONS STOCK AND WATSON (1991) MODEL; NO REGIME SWITCHING

	Prior		Posterior			
	Mean	SD	Mean	SD	MD	95% Bands
ΔC_t						
ϕ_1	0.0	1	0.533	0.069	0.534	(0.395, 0.669)
ϕ_2	0.0	1	0.029	0.062	0.029	(−0.091, 0.153)
δ	—	—	0.567	0.046	0.566	(0.483, 0.662)
Δy_{1t}						
λ_1	0.0	1	0.621	0.041	0.619	(0.544, 0.705)
ψ_{11}	0.0	1	−0.032	0.065	−0.036	(−0.163, 0.095)
ψ_{12}	0.0	1	−0.054	0.065	−0.055	(−0.176, 0.074)
σ_1^2	—	—	0.236	0.034	0.235	(0.173, 0.303)
Δy_{2t}						
λ_2	0.0	1	0.231	0.023	0.231	(0.188, 0.278)
ψ_{21}	0.0	1	−0.302	0.052	−0.303	(−0.402, −0.202)
ψ_{22}	0.0	1	−0.065	0.051	−0.066	(−0.163, 0.033)
σ_2^2	—	—	0.316	0.023	0.315	(0.274, 0.363)
Δy_{3t}						
λ_3	0.0	1	0.479	0.038	0.477	(0.406, 0.556)
ψ_{31}	0.0	1	−0.358	0.053	−0.356	(−0.460, −0.253)
ψ_{32}	0.0	1	−0.153	0.053	−0.153	(−0.256, −0.051)
σ_3^2	—	—	0.668	0.052	0.666	(0.571, 0.775)
Δy_{4t}						
λ_{40}	0.0	1	0.133	0.010	0.133	(0.113, 0.153)
λ_{41}	0.0	1	0.003	0.011	0.003	(−0.018, 0.024)
λ_{42}	0.0	1	0.024	0.010	0.024	(0.004, 0.043)
λ_{43}	0.0	1	0.028	0.008	0.028	(0.011, 0.045)
ψ_{41}	0.0	1	−0.003	0.060	−0.002	(−0.120, 0.110)
ψ_{42}	0.0	1	0.293	0.060	0.294	(0.170, 0.410)
σ_4^2	—	—	0.021	0.002	0.021	(0.017, 0.026)

Notes: (1) y_{1t}, y_{2t}, y_{3t}, and y_{4t} represent *IP*, *GMYXPQ*, *MTQ*, and *LPNAG*, respectively.
(2) Prior distribution of σ_i^2 is improper.
(3) SD and MD refer to standard deviation and median, respectively.
(4) 95% bands refers to 95% posterior probability bands.

FIGURE 5.—NEW COINCIDENT INDEX VERSUS STOCK AND WATSON'S COINCIDENT INDEX



regime switching. Overall, the Stock and Watson index agrees more closely with the DOC index than our new experimental index.

IV. Tests of Business Cycle Duration Dependence

Duration dependence in the business cycle and time-varying transition probabilities are related but not identical concepts. Duration dependence asks whether recessions or booms "age," that is, are more likely to end as they last longer. It is a special form of time-varying transition probabilities in a regime-switching model of the business cycle. Diebold and Rudebusch (1990) and Diebold et al. (1993) find evidence that postwar recessions tend to have positive duration dependence, although booms do not. More recently, Durland and McCurdy (1994), using Hamilton's (1989) univariate Markov-switching model of business cycle, reach the same conclusion. Diebold et al. (1994), Filardo (1994), and Filardo and Gordon (1993) deal with another form of time-varying transition probabilities, specifying them as functions of an exogenous variable such as the index of leading indicators. None of these papers specifically deals with duration dependence in a multivariate context. In this section we investigate the hypothesis of duration dependence by extending the dynamic factor model in section II to incorporate nonzero probabilities of duration dependence in the transition probabilities.

A. *Extension of the Model to Allow for Nonzero Probabilities of Business Cycle Duration Dependence*

To construct a formal test of business cycle duration dependence, the transition probabilities in equation (5) are replaced by the following probit specification for the evolution of the business cycle regime variable S_t:

$$\Pr[S_t = 1] = \Pr[S_t^* \geq 0] \tag{12}$$

where S_t^* is a latent variable defined as

$$\begin{aligned} S_t^* = \gamma_0 + \gamma_1 S_{t-1} + \gamma_{2,d_2}(1 - S_{t-1})N_{0,t-1} \\ + \gamma_{3,d_3} S_{t-1} N_{1,t-1} + u_t \end{aligned} \tag{13}$$

$$d_2 = 0 \text{ or } 1, \qquad d_3 = 0 \text{ or } 1 \tag{14}$$

$$u_t \sim \text{iid } N(0, 1) \tag{15}$$

where $N_{j,t-1}$, $j = 0, 1$ are the durations up to time $t - 1$ of a recession or boom, respectively; and the parameters γ_{2,d_2} and γ_{3,d_3}, which will be discussed in detail later, determine the nature of business cycle duration dependence. In the above probit specification, the transition probabilities are given by:

$$\begin{aligned} &\Pr[S_t = 1 | S_{t-1} = 1, N_{1,t-1}] \\ &\quad = \Pr[S_t^* \geq 0 | S_{t-1} = 1, N_{1,t-1}] \\ &\quad = \Pr[u_t \geq -\gamma_0 - \gamma_1 - \gamma_{3,d_3} N_{1,t-1}] \end{aligned} \tag{16}$$

$$\begin{aligned} &\Pr[S_t = 0 | S_{t-1} = 0, N_{0,t-1}] \\ &\quad = \Pr[S_t^* < 0 | S_{t-1} = 0, N_{0,t-1}] \\ &\quad = \Pr[u_t < -\gamma_0 - \gamma_{2,d_2} N_{0,t-1}]. \end{aligned} \tag{17}$$

When $\gamma_{2,d_2} = 0$ and $\gamma_{3,d_3} = 0$, we have fixed transition probabilities or no business cycle duration dependence; when $\gamma_{2,d_2} > 0$ and $\gamma_{3,d_3} < 0$, business cycles are characterized by positive duration dependence. In order to allow for nonzero probabilities of positive duration dependence, we adopt assumptions similar to those introduced in George and McCulloch (1993) and Geweke (1994) in the context of variable selection in regression. In particular, while we assume γ_0 and γ_1 each are nonzero with prior probability 1, we follow Geweke (1994) in adopting a prior for γ_{2,d_2} and γ_{3,d_3} that is a mixture of a truncated normal and point mass at 0;

$$\gamma_{2,d_2} \begin{cases} = 0, & \text{if } d_2 = 0 \\ \sim N(\underline{\gamma}_2, \underline{\omega}_2^2)_{I[\gamma_{2,d_2} > 0]}, & \text{if } d_2 = 1 \end{cases} \tag{18}$$

$$\gamma_{3,d_3} \begin{cases} = 0, & \text{if } d_3 = 0 \\ \sim N(\underline{\gamma}_3, \underline{\omega}_3^2)_{I[\gamma_{3,d_3} < 0]}, & \text{if } d_3 = 1 \end{cases} \tag{19}$$

$$\Pr[d_2 = 0] = \Pr[\gamma_{2,d_2} = 0] \equiv p_2 \tag{20}$$

$$\Pr[d_3 = 0] = \Pr[\gamma_{3,d_3} = 0] \equiv p_3 \tag{21}$$

TABLE 4.—SENSITIVITIES OF POSTERIOR PROBABILITIES OF NO DURATION DEPENDENCE TO DIFFERENT PRIORS
$\underline{\omega}_0 = \underline{\omega}_1 = 0.3$; $\underline{\omega}_2 = \underline{\omega}_3 = 0.05$

	Pr $[d_2 = 0]$: *For Recessions*					
Prior ($p_2 = p_3$)	0.300	0.400	0.500	0.600	0.700	0.800
Posterior	0.014	0.023	0.033	0.044	0.070	0.113
	Pr $[d_3 = 0]$: *For Booms*					
Prior ($p_2 = p_3$)	0.300	0.400	0.500	0.600	0.700	0.800
Posterior	0.122	0.196	0.276	0.355	0.460	0.595
	Pr $[d_2 = 0, d_3 = 0]$: *For Joint Tests*					
Prior ($p_2 = p_3$)	0.300	0.400	0.500	0.600	0.700	0.800
Posterior	0.001	0.004	0.008	0.015	0.028	0.054

where the subscript $I[\cdot]$ refers to an indicator function introduced to allow for positive business cycle duration dependence under the alternative hypothesis; and p_2 and p_3 are independent prior probabilities of no duration dependence for recessions and booms. Each of these prior distributions includes the possibility that the duration variables $N_{0,t-1}$ and $N_{1,t-1}$ are excluded from the model. Given these priors, Bayesian tests of business cycle duration dependence are to read the posterior probabilities of no duration dependence directly from the proportion of posterior simulations in which $d_2 = 0$ and $d_3 = 0$.

B. *Implementation of Gibbs Sampling*

Replacing the fixed transition probabilities in equation (5) by potentially duration-dependent transition probabilities in equations (16) and (17) requires modifications of the basic Gibbs sampler described in appendix A. Specifically, the procedure for generating S_t, $t = 1, 2, \ldots, T$, in appendix A.2 and that for generating the transition probabilities in appendix A.6 should be modified.

Due to the time-varying nature of the transition probabilities, S_t, $t = 1, 2, \ldots, T$, cannot be generated as a block as in appendix A.2. Each S_t should be generated one at a time conditional on $S_{j\neq t}$, $j = 1, 2, \ldots, T$, and on other variates. It is straightforward to modify Albert and Chib's (1993) procedure to achieve this goal. Once S_t, $t = 1, 2, \ldots, T$, is generated, we can count the duration of a recession or a boom ($N_{1,t-1}$ or $N_{0,t-1}$) up to month $t-1$, $t = 2, 3, \ldots, T$. Also, given γ_0, γ_1, γ_{2,d_2}, γ_{3,d_3}, each generated set $\tilde{S}_T$ can be converted to a set of latent variables $\tilde{S}^*_T = [S^*_1 \quad S^*_2 \quad \cdots \quad S^*_T]'$ based on equation (13), by generating u_t from an appropriate truncated standard normal distribution. Conditional on $S_t = 1$ and $S_{t-1} = 1$, for example, u_t is generated from

$$u_t \sim N(0, 1)_{I[u_t \geq -(\gamma_0+\gamma_1+\gamma_{3,d_3}N_{1,t-1})]} \tag{22}$$

where the subscript $I[\cdot]$ refers to an indicator function.

Conditional on $\tilde{S}_T = [S_1 \quad S_2 \quad \cdots \quad S_T]'$ and, thus, on $\tilde{S}^*_T = [S^*_2 \quad S^*_3 \quad \cdots \quad S^*_T]'$, $\tilde{N}_{1,-1} = [N_{1,1} \quad N_{1,2} \quad \cdots \quad N_{1,T-1}]'$, and $\tilde{N}_{0,-1} = [N_{0,1} \quad N_{0,2} \quad \cdots \quad N_{0,T-1}]'$, the variates in equation (13) are independent of the data set and other variates in the system. This nice conditioning feature of the Gibbs sampling allows us to focus on equation (13) for generating transition probabilities and other related parameters for tests of business cycle duration dependence. Thus the transition probabilities can be generated directly from equations (16) and (17). Appendix C describes in detail how d_i, $i = 2, 3$, and γ variables in equation (13) can be generated. Generating the other variates in the system is exactly the same as in the case of fixed transition probabilities, as described in appendix A.

TABLE 5.—SENSITIVITIES OF POSTERIOR PROBABILITIES OF NO DURATION DEPENDENCE TO DIFFERENT PRIORS
$\underline{\omega}_0 = \underline{\omega}_1 = 0.3$; $\underline{\omega}_2 = \underline{\omega}_3 = 0.1$

	Pr $[d_2 = 0]$: *For Recessions*					
Prior ($p_2 = p_3$)	0.300	0.400	0.500	0.600	0.700	0.800
Posterior	0.029	0.042	0.067	0.081	0.114	0.187
	Pr $[d_3 = 0]$: *For Booms*					
Prior ($p_2 = p_3$)	0.300	0.400	0.500	0.600	0.700	0.800
Posterior	0.405	0.533	0.637	0.734	0.767	0.874
	Pr $[d_2 = 0, d_3 = 0]$: *For Joint Tests*					
Prior ($p_2 = p_3$)	0.300	0.400	0.500	0.600	0.700	0.800
Posterior	0.010	0.018	0.040	0.056	0.080	0.162

TABLE 6.—SENSITIVITIES OF POSTERIOR PROBABILITIES OF NO DURATION DEPENDENCE TO DIFFERENT PRIORS
$\underline{\omega}_0 = \underline{\omega}_1 = 0.1$; $\underline{\omega}_2 = \underline{\omega}_3 = 0.1$

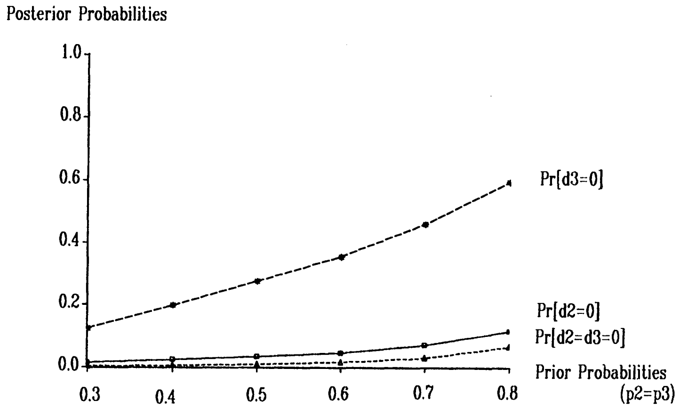
	Pr $[d_2 = 0]$: *For Recessions*					
Prior ($p_2 = p_3$)	0.300	0.400	0.500	0.600	0.700	0.800
Posterior	0.020	0.032	0.036	0.049	0.089	0.153
	Pr $[d_3 = 0]$: *For Booms*					
Prior ($p_2 = p_3$)	0.300	0.400	0.500	0.600	0.700	0.800
Posterior	0.482	0.549	0.673	0.802	0.807	0.876
	Pr $[d_2 = 0, d_3 = 0]$: *For Joint Tests*					
Prior ($p_2 = p_3$)	0.300	0.400	0.500	0.600	0.700	0.800
Posterior	0.008	0.017	0.021	0.036	0.067	0.123

C. *Empirical Results*

Of particular interest in this section are the proportions of posterior simulations in which (1) $d_2 = 0$ for tests of no duration dependence for recessions; (2) $d_3 = 0$ for a test of no duration dependence for booms; and (3) $d_2 = d_3 = 0$ for a joint test of no business cycle duration dependence. We adopt different prior probabilities ($p_2 = p_3 = 0.3$, 0.4, 0.5, 0.6, 0.7, 0.8) of no duration dependence and investigate sensitivities of posterior probabilities to these priors. We also adopt different priors for γ_0, γ_1, $\gamma_{2,1}$, and $\gamma_{3,1}$ in equation (13) and investigate sensitivities of the results.[10] Prior distributions employed for these parametes are $\gamma_0 \sim N(-1.28, \underline{\omega}_0^2)$ and $\gamma_1 \sim N(3.24, \underline{\omega}_1^2)$ (these are equivalent to assuming average durations of recessions and booms to be 10 months and 33.3 months under the null hypothesis of

[10] As mentioned in George and McCulloch (1993), from a subjectivist Bayesian standpoint, one would want to choose $\underline{\omega}_j^2$, $j = 2, 3$, large enough to give support to values of $\gamma_{j,1}$, $j = 2, 3$, that are substantively different from 0, but not so large that unrealistic values of $\gamma_{j,1}$, $j = 2, 3$, are supported.

FIGURE 6.—SENSITIVITIES OF POSTERIOR PROBABILITIES OF NO DURATION DEPENDENCE TO DIFFERENT PRIOR PROBABILITIES
$\omega_0 = \omega_1 = 0.3; \omega_2 = \omega_3 = 0.05$



no duration dependence, as in section III); $\gamma_{2,1} \sim N(0, \underline{\omega}_2^2)_{I[\gamma_{2,1}>0]}$ and $\gamma_{3,1} \sim N(0, \underline{\omega}_3^2)_{I[\gamma_{3,1}<0]}$. We try the following three different combinations of $\underline{\omega}$:

(1) *Case 1:* $\underline{\omega}_0 = \underline{\omega}_1 = 0.3$ and $\underline{\omega}_2 = \underline{\omega}_3 = 0.05$.
(2) *Case 2:* $\underline{\omega}_0 = \underline{\omega}_1 = 0.3$ and $\underline{\omega}_2 = \underline{\omega}_3 = 0.1$.
(3) *Case 3:* $\underline{\omega}_0 = \underline{\omega}_1 = 0.1$ and $\underline{\omega}_2 = \underline{\omega}_3 = 0.1$.

Throughout these experiments, we maintain the priors of all the other variates in the system to be the same as in section III (tables 1 and 2). For each of the above 18 different sets of prior distributions, we run the Gibbs sampler for 12,000 iterations. For most of the cases the Gibbs sampler seems to converge in less than 3000 iterations. To be on the safe side, we discard the first 4000 and base our inferences on the last 8000 iterations.

Tables 4 through 6 summarize the sensitivities of posterior probabilities of no duration dependence to different priors. These are also depicted in figures 6 through 8. In general, smaller values of $\underline{\omega}_j$, the prior standard deviations of γ, the parameters result in lower posterior probabilities

FIGURE 7.—SENSITIVITIES OF POSTERIOR PROBABILITIES OF NO DURATION DEPENDENCE TO DIFFERENT PRIOR PROBABILITIES
$\omega_0 = \omega_1 = 0.3; \omega_2 = \omega_3 = 0.1$

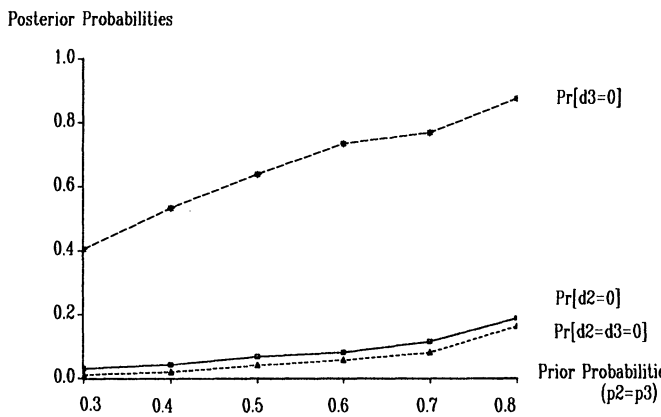
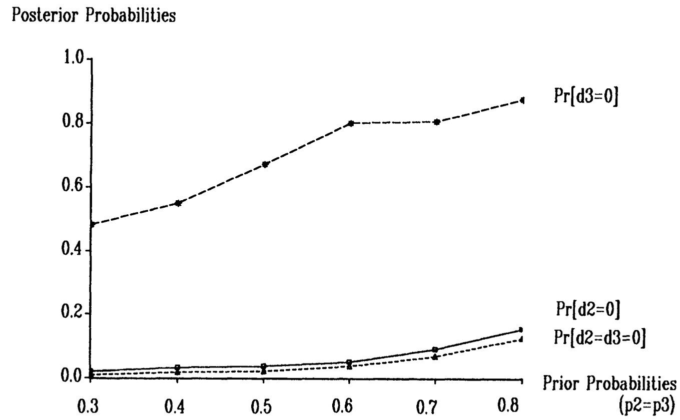


FIGURE 8.—SENSITIVITIES OF POSTERIOR PROBABILITIES OF NO DURATION DEPENDENCE TO DIFFERENT PRIOR PROBABILITIES
$\omega_0 = \omega_1 = 0.1; \omega_2 = \omega_3 = 0.1$



for given prior probabilities. Concerning tests of no duration dependence for recessions, the results are robust with respect to different priors: posterior probabilities range between 0.014 and 0.187, revealing strong sample information in favor of positive duration dependence beyond that contained in the prior probabilities. Concerning tests of no duration dependence for booms, results for case 1 tend to reveal somewhat weak sample information in favor of positive duration dependence: posterior probabilities of no duration dependence in case 1 range between 0.122 and 0.595. Results from the other two cases, however, reveal no such sample information beyond that contained in the prior probabilities.

V. Summary and Conclusions

This paper estimates a model that incorporates the two key features of business cycles identified by Burns and Mitchell (1946): comovement among economic variables and switching between distinct regimes of recession and boom. A common factor, interpreted as a composite index of coincident indicators and estimates of turning points from one regime to the other were extracted from the four monthly DOC coincident indicators, utilizing the multimove Gibbs sampling of Shephard (1994) and Carter and Kohn (1994) in a Bayesian framework. The empirical results demonstrate that both comovement among indicators and regime switching are important features of the business cycle. The resulting model-based index of coincident indicators provides a somewhat different picture of the strength of recent business cycles than does the DOC index. Estimates of turning points are sharper and agree much more closely with the NBER dates than do estimates from a univariate model.

The approach in this paper provides us with more information than may be extracted from maximum-likelihood estimation of the model. In particular, a straightforward extension of the base model leads naturally to Bayesian tests of business cycle duration dependence. By taking advantage of ideas set forth in Geweke (1994) in the context of variable selection in regression, we extend the base model to include nonzero probabilities of business cycle duration dependence.

While existing literature (Diebold and Rudebusch (1990), Diebold et al. (1993), and Durland and McCurdy (1994), among others) report evidence of positive duration dependence for recessions and little evidence for booms within a univariate context, we confirm their results within a multivariate context. We find strong evidence of positive duration dependence for recessions, and the results are robust with respect to different priors employed. As for booms, however, the results are not robust with respect to different priors employed. Depending on the choice of priors, posterior probabilities of no duration dependence tend to reveal a certain degree of sample information in favor of positive duration dependence for booms. However, the evidence is not strong enough.

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APPENDIX A

Gibbs Sampling for a Model with Fixed Transition Probabilities

A.1 Generating Δc_t, $t = 1, 2, \ldots, T$

As mentioned earlier, there is more than one way of casting the model in equations (1)–(2) or (8)–(9) into state–space form. For the present purpose we employ an alternative state–space representation to the one in equations (10)–(11).[11] If we multiply both sides of equation (8) by $\psi_i(L)$, $i = 1, 2, 3, 4$, we have $\Delta y_{it}^* = \lambda_i(L)\psi_i(L)\Delta c_t + \epsilon_{it}$, $i = 1, 2, 3, 4$, where $\Delta y_{it}^* =$

[11] The state–space representation in equations (10)–(11) is used later when decomposing $\Delta \bar{Y}_i$ into D_i and δ based on the steady-state Kalman gain.

$\psi_i(L)\Delta y_{it}$, resulting in the following alternative state–space representation:

$$\Delta y_t^* = H^* \zeta_t + \epsilon_t \tag{10'}$$

$$\zeta_t = \tilde{M}_{s_t}^* + F^* \zeta_{t-1} + u_t^*. \tag{11'}$$

Assuming AR(1) common and idiosyncratic components and for $\lambda_i(L) = \lambda_i$, for example, we have

$$\begin{bmatrix} \Delta y_{1t}^* \\ \Delta y_{2t}^* \\ \Delta y_{3t}^* \\ \Delta y_{4t}^* \end{bmatrix} = \begin{bmatrix} \lambda_1 & -\lambda_1\psi_{11} \\ \lambda_2 & -\lambda_2\psi_{21} \\ \lambda_3 & -\lambda_3\psi_{31} \\ \lambda_4 & -\lambda_4\psi_{41} \end{bmatrix} \begin{bmatrix} \Delta c_t \\ \Delta c_{t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \\ \epsilon_{3t} \\ \epsilon_{4t} \end{bmatrix} \tag{10''}$$

$$\begin{bmatrix} \Delta c_t \\ \Delta c_{t-1} \end{bmatrix} = \begin{bmatrix} \phi(L)\mu_{s_t} \\ 0 \end{bmatrix} + \begin{bmatrix} \phi_1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta c_{t-1} \\ \Delta c_{t-2} \end{bmatrix} + \begin{bmatrix} v_t \\ 0 \end{bmatrix}. \tag{11''}$$

Now consider the following joint distribution of $\tilde{\zeta}_T = [\zeta_1 \;\; \zeta_2 \;\; \zeta_3 \;\; \cdots \;\; \zeta_T]'$, given $\Delta\tilde{y}_T^* = [\Delta y_1^* \;\; \Delta y_2^* \;\; \cdots \;\; \Delta y_T^*]'$ and given the prior distribution of ζ_0:

$$p(\tilde{\zeta}_T|\Delta\tilde{y}_T^*) = p(\zeta_T|\Delta\tilde{y}_T^*) \prod_{t=1}^{T-1} p(\zeta_t|\Delta\tilde{y}_t^*, \zeta_{t+1}). \tag{A.1}$$

If we pay attention to the time ordering of the state–space model as in Carter and Kohn (1994), generation of $\tilde{\zeta}_T$ from $p(\tilde{\zeta}_T|\,\tilde{y}_T^*)$ can be done in the following sequence. First, we can generate ζ_T from $p(\zeta_T|\Delta\tilde{y}_T^*)$. Then, for $t = T-1, T-2, \ldots, 1$, we can generate ζ_t from $p(\zeta_t|\Delta\tilde{y}_t^*, \zeta_{t+1})$. As the state–space model in equations (10)′–(11)′ is linear and Gaussian, conditional on S_t, $t = 1, 2, \ldots, T$, we can take advantage of the Gaussian Kalman filter to obtain $p(\zeta_T|\Delta\tilde{y}_T^*)$ and $p(\zeta_t|\Delta\tilde{y}_t^*, \zeta_{t+1})$.

Notice also that all other elements of ζ_t except for the first element, Δc_t, are associated with elements of ζ_{t-1} by the identity matrix. The joint density in equation (A.1) can therefore be rewritten alternatively in terms of the first element of ζ_t, Δc_t,

$$\begin{aligned} p(\tilde{\zeta}_T|\Delta\tilde{y}_T^*) &= p(\Delta\tilde{c}_T|\Delta\tilde{y}_T^*) \\ &= p(\Delta c_T|\tilde{\Delta}y_T^*) \prod_{t=1}^{T-1} p(\Delta c_t|\Delta\tilde{y}_t^*, \Delta c_{t+1}). \end{aligned} \tag{A.2}$$

Keeping these relationships in mind, we can proceed to generate Δc_t, $t = 1, 2, \ldots, T$, as follows.

Step 1: Run the Kalman filter algorithm to calculate $\zeta_{t|t} = E(\zeta_t|\Delta\tilde{y}_t^*)$ and $V_{t|t} = \text{var}(\zeta_t|\Delta\tilde{y}_t^*)$ for $t = 1, 2, \ldots, T$ and save them. The last iteration of the Kalman filter provides us with $\zeta_{T|T}$ and $V_{T|T}$, and the (1, 1) elements of $\zeta_{T|T}$ and $V_{T|T}$ can be used to generate Δc_T.

Step 2: For $t = T-1, T-2, \ldots, 1$, given $\zeta_{t|t}$ and $V_{t|t}$, if we treat the generated Δc_{t+1} as an additional observation to the system, the distribution $p(\zeta_t|\Delta\tilde{y}_t^*, \Delta c_{t+1})$ is easily derived by applying the updating equations of the Kalman filter. As Δc_{t+1}, the first element of ζ_{t+1}, is given by

$$\Delta c_{t+1} = \phi(L)\mu_{s_{t+1}} + F^*(1)\zeta_t + \mu_{t+1}^*(1) \tag{A.3}$$

where $F^*(1)$ is the first row of F^* and $u_{t+1}^*(1)$ is the first element of u_{t+1}^*, updating equations are derived as

$$\zeta_{t|t,\Delta c_{t+1}} = \zeta_{t|t} + V_{t|t}F^*(1)\eta_t/R_t \tag{A.4}$$

$$V_{t|t,\Delta c_{t+1}} = V_{t|t} - V_{t|t}F^*(1)'F^*(1)V_{t|t}/R_t \tag{A.5}$$

where $\eta_t = \Delta c_{t+1} - \phi(L)\mu_{s_{t+1}} - F^*(1)\zeta_{t|t}$ and $R_t = F^*(1)V_{t|t}F^*(1)' + \text{var}\,[u_{t+1}(1)]$. Then the (1, 1) elements of $\zeta_{t|t,\Delta c_{t+1}}$ and $V_{t|t,\Delta c_{t+1}}$ can be used to generate Δc_t, for $t + T - 1, T - 2, \ldots, 1$.

A.2 *Generating* S_t, $t = 1, 2, \ldots, T$

Once $\Delta\tilde{c}_T = [\Delta c_1 \;\; \cdots \;\; \Delta c_T]'$ has been generated, $\tilde{S}_T = [S_1 \;\; \cdots \;\; S_T]'$ can be generated based on the following distribution:

$$\begin{aligned} p(\tilde{S}_T|\Delta\tilde{y}_T^*, \Delta\tilde{c}_T) &= p(S_T|\Delta\tilde{y}_T^*, \Delta\tilde{c}_T) \prod_{t=1}^{T-1} p(S_t|\Delta\tilde{y}_t^*, \Delta\tilde{c}_t, S_{t+1}) \\ &= p(S_T|\Delta\tilde{c}_T) \prod_{t=1}^{T-1} p(S_t|\Delta\tilde{c}_t, S_{t+1}). \end{aligned} \tag{A.6}$$

This implies that we may focus only on equation (9) to generate $\tilde{S}_T$, as the distribution of $\tilde{S}_T$ is independent of $\Delta\tilde{y}_T^*$, given $\Delta\tilde{c}_T$. To generate $\tilde{S}_t$, we can adopt the following steps.

Step 1: Run Hamilton's (1989) basic filter for equation (9) to get $p(S_t|\Delta\tilde{c}_t)$ and $p(S_t|\Delta\tilde{c}_{t-1})$ for $t = 1, 2, \ldots, T$ and save them. The last iteration of the filter provides us with $p(S_T|\Delta\tilde{c}_T)$, from which S_T is generated.

Step 2: To generate S_t conditional on $\Delta\tilde{c}_t$ and S_{t+1}, $t = T-1, T-2, \ldots, 1$, we employ the following result:

$$p(S_t|\Delta\tilde{c}_t, S_{t+1}) = \frac{p(S_{t+1}|S_t)p(S_t|\Delta\tilde{c}_t)}{p(S_{t+1}|\Delta\tilde{c}_t)} \tag{A.7}$$

where $P(S_{t+1}|S_t)$ is the transition probability, and the other terms on the right-hand side have been saved from step 1.

A.3 *Generating* λ_i, $i = 1, 2, 3, 4$[12]

Given $\Delta\tilde{c}_T$, $\tilde{S}_T$, and all other parameters of the model, λ_i can be generated based on equation (8) multiplied by $\psi_i(L)$, $i = 1, 2, 3, 4$,

$$y_{it}^* = \lambda_i \Delta c_t^* + \epsilon_{it}, \quad i = 1, 2, 3, 4 \tag{A.8}$$

where $y_{it}^* = \psi_i(L)\Delta y_{it}$ and $\Delta c_t^* = \psi_i(L)\Delta c_t$. Defining $\Delta\tilde{y}_i^*$ and $\Delta\tilde{C}^*$ to be the vectors of left-hand-side and right-hand-side variables, respectively, in equation (A.8), λ_i can be generated from the following posterior distributions:

$$\begin{aligned} \lambda_i \sim N(&(A_i^{-1} + \sigma_i^{-2}\Delta\tilde{C}^{*\prime}\Delta\tilde{C}^*)^{-1}(A_i^{-1}a_i + \sigma_i^{-2}\Delta\tilde{C}^{*\prime}\Delta\tilde{y}_i^*), \\ &(A_i^{-1} + \sigma_i^{-2}\Delta\tilde{C}^{*\prime}\Delta\tilde{C}^*)^{-1}), \quad i = 1, 2, 3, 4 \end{aligned} \tag{A.9}$$

where the prior distributions for λ_i, $i = 1, 2, 3, 4$, are given by

$$\lambda_i \sim N(a_i, A_i), \qquad i = 1, 2, 3, 4. \tag{A.10}$$

A.4 *Generating* $\tilde{\psi}_i$ *and* σ_i^2, $i = 1, 2, 3, 4$

Given $\Delta\tilde{c}_t$ and λ_t, equation (8) can be rewritten as

$$\Delta y_{it} - \lambda_i \Delta c_t = e_{it}, \qquad i = 1, 2, 3, 4. \tag{A.11}$$

By multiplying both sides of equation (A.11) by $\psi_i(L)$, we get

$$\psi_i(L)Z_t = \epsilon_{it}, \qquad i = 1, 2, 3, 4 \tag{A.12}$$

[12] For expositional purpose we assume that $\lambda_i(L) = \lambda_i$, $i = 1, 2, 3, 4$.

or assuming that $\psi_i(L) = 1 - \psi_{i1}L - \cdots - \psi_{im}L^m$

$$Z_t = \psi_{i1}Z_{t-1} + \psi_{i2}Z_{t-1} + \cdots + \psi_{im}Z_{t-m} + \epsilon_{it}, \quad i = 1, 2, 3, 4 \tag{A.13}$$

where $Z_t = \Delta y_{it} - \lambda_i \Delta c_t$. Defining $\tilde{Z}$ to be the vector of the Z_t, and $\tilde{X}$ to be the matrix of the right-hand-side variables, the posterior distribution of $\tilde{\psi}_i = [\psi_{i1} \ \ldots \ \psi_{im}]'$ is given by

$$\tilde{\psi}_i \sim N\left(\left(\prod_i^{-1} + \sigma_i^{-2}\tilde{X}'\tilde{X}\right)^{-1}\left(\prod_i^{-1}\pi_i + \sigma_i^{-2}\tilde{X}'\tilde{Z}\right), \left(\prod_i^{-1} + \sigma_i \tilde{X}'\tilde{X})^{-1}\right)\right), \quad i = 1, 2, 3, 4 \tag{A.14}$$

where the prior distribution for $\tilde{\psi}_i$ has the multivariate normal form

$$\tilde{\psi}_i \sim N\left(\pi_i, \prod_i\right), \quad i = 1, 2, 3, 4. \tag{A.15}$$

Using the above posterior distribution of $\tilde{\psi}_i$, values of $\tilde{\psi}_i$ can be generated, given σ_i^2, $i = 1, 2, 3, 4$. Then, given the generated values of $\tilde{\psi}_i$, the following posterior distributions of σ_i^2 can be employed to generate σ_i^2:

$$\sigma_i^2 \sim \text{IG}\left(\frac{\nu_i + T}{2}, \frac{f_i}{2} + \frac{1}{2}(\tilde{Z} - \tilde{X}\tilde{\psi}_i)'(\tilde{Z} - \tilde{X}\tilde{\Psi}_i)\right), \quad i = 1, 2, 3, 4 \tag{A.16}$$

where IG denotes the inverse gamma distribution, the prior distribution is given by IG $(\nu_i/2, f_i/2)$, and the hyperparameters ν_i and f_i are known. ν reflects the strength of the prior of σ_e^2.

A.5 Generating μ_0, μ_1, and $\tilde{\phi}$

Given the generated values of $\Delta\tilde{c}_T$ and $\tilde{S}_T$, the last step of the Gibbs sampling algorithm is to generate parameters in equation (9). A convenient conditioning feature of the Gibbs sampling approach is that it makes equation (9) linear, conditional on $\tilde{S}_t$.

We first generate $\tilde{\phi} = [\phi_1 \ \ldots \ \phi_n]'$, given μ_0 and μ_1. Rewriting equation (9) for the case of an AR (n) process, we have

$$(\Delta c_t - \mu_{s_t}) = \phi_1(\Delta c_{t-1} - \mu_{s_{t-1}}) - \cdots - \phi_n(\Delta c_{t-n} - \mu_{s_{t-n}}) + v_t. \tag{A.17}$$

Letting $\tilde{G}$ denote the vector of the left-hand-side variables and $\tilde{Q}$ the matrix of the right-hand-side variables, the multivariate normal distribution from which $\tilde{\phi}$ is generated is given by

$$\tilde{\phi} \sim N((A^{-1} + \tilde{Q}'\tilde{Q})^{-1}(A^{-1}\alpha + \tilde{Q}'\tilde{G}), (A^{-1} + \tilde{Q}'\tilde{Q})^{-1}) \tag{A.18}$$

where the prior distribution for $\tilde{\phi}$ has the multivariate normal form

$$\tilde{\phi} \sim N(\alpha, A). \tag{A.19}$$

Given $\tilde{\phi}$, equation (9) can again be rewritten as

$$(\Delta c_t - \phi_1 \Delta c_{t-1} - \cdots - \phi_n \Delta c_{t-n}) = \mu_0^* + \mu_1(S_t - \phi_1 S_{t-1} - \cdots - \phi_n S_{t-n}) + v_t \tag{A.20}$$

where $\mu_0^* = \mu_0(1 - \phi_1 - \cdots - \phi_p)$. Letting $\tilde{G}^*$ be the vector of left-hand-side variables and $\tilde{Q}^*$ the matrix of right-hand-side variables, $\tilde{\mu} = [\mu_0^* \ \ \mu_1]$ can be generated from the following multivariate distribution:

$$\tilde{\mu} \sim N((A^{*-1} + \tilde{Q}^{*\prime}\tilde{Q}^*)^{-1}(A^{*-1}\alpha^* + \tilde{Q}^{*\prime}\tilde{G}^*), (A^{*-1} + \tilde{Q}^{*\prime}\tilde{Q}^*)^{-1})_{I[\mu_1>0]} \tag{A.21}$$

where the subscript $I[\cdot]$ refers to the indicator function on $[\mu_1 > 0]$, and where the prior distribution for $\tilde{\mu}$ has the multivariate normal form

$$\tilde{\mu} \sim N(\alpha^*, A^*)_{I[\mu_1>0]}. \tag{A.22}$$

Values of μ can therefore be simulated from the above truncated multivariate normal distribution.

A.6 Generating Transition Probabilities p *and* q

We follow Albert and Chib (1993) in deriving the conditional distribution of p and q. Given $\tilde{S}_T$ and the initial state, let the transitions from state $S_{t-1} = i$ to $S_t = j$ be denoted by n_{ij}. Then the likelihood function is given by

$$L(q, p) = q^{n_{00}}(1 - q)^{n_{01}}p^{n_{11}}(1 - p)^{n_{10}}. \tag{A.23}$$

We therefore take the independent beta family of distributions as a conjugate prior for each of the transition probabilities

$$\pi(q, p) \propto q^{u_{00}}(1 - q)^{u_{01}}p^{u_{11}}(1 - p)^{u_{10}}. \tag{A.24}$$

Combining the likelihood function and the prior, we get the conditional distribution of $[q, p]$ as a product of the following independent beta distributions, from which we can generate p and q:

$$[q|\tilde{S}_T] \sim \beta(u_{00} + n_{00}, u_{01} + n_{01}) \tag{A.25}$$

$$[p|\tilde{S}_T] \sim \beta(u_{11} + n_{11}, u_{10} + n_{10}). \tag{A.26}$$

A.7 Calculation of the Composite Coincident Index

The mean of each coincident variable $(\Delta w\overline{Y}_i)$ consists of the portion due to the idisyncratic component [D_i in equation (1)] and the portion due to the common component [δ in equation (2)]. We now discuss how we might decompose $\Delta\overline{Y}_i$, $i = 1, 2, 3, 4$, into its two components.

As mentioned earlier, conditional on $\tilde{S}_T$, the state-space models given by equations (10)–(11) and (10)′–(11)′ are linear and Gaussian. The method of decomposing $\Delta\overline{Y}_i$ is therefore readily available, as shown in Stock and Watson (1991). At each run of Gibbs sampling, conditional on all the parameters of the model and $\tilde{S}_T$ as generated from equations (10)′ and (11)′, we apply the Kalman filter recursion to equations (10) and (11). Denoting by K^* the steady-state Kalman gain from the filter, the mean of each series $\Delta\overline{Y}_i$ is decomposed into the two components in the following way:[13]

$$\delta = E_1'[K_k - (I_k - K^*H)F]^{-1}K^*\Delta\overline{Y} \tag{A.27}$$

where $E_1' = [1 \ 0 \ 0 \ \cdots \ 0]$, H is the selection matrix in the measurement equation (10), F is the transition matrix in equation (11), k is the dimension of the F matrix, and $\Delta\overline{Y} = [\Delta\overline{Y}_1 \ \ \Delta\overline{Y}_2 \ \ \Delta\overline{Y}_3 \ \ \Delta\overline{Y}_4]'$. Once δ is identified, for given $\Delta\tilde{c}_T$, and an initial value of C_0, we are able to generate our new composite coincident index[14] $C_t = \Delta c_t + C_{t-1} + \delta$, $t = 1, 2, \ldots, T$.

[13] For details, refer to Stock and Watson (1991, pp. 70–71).

[14] The index is unitless and identified up to an arbitrary choice of C_0.

APPENDIX B

State–Space Representations of the Empirical Model

B.1 State–Space Representation 1

(1) Measurement equation:

$$\begin{bmatrix}\Delta y_{1t}\\ \Delta y_{2t}\\ \Delta y_{3t}\\ \Delta y_{4t}\end{bmatrix} = \begin{bmatrix}\lambda_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ \lambda_2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ \lambda_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ \lambda_{40} & \lambda_{41} & \lambda_{42} & \lambda_{43} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix} \begin{bmatrix}\Delta c_t\\ \Delta c_{t-1}\\ \Delta c_{t-2}\\ \Delta c_{t-3}\\ e_{1t}\\ e_{1,t-1}\\ e_{2t}\\ e_{2,t-1}\\ e_{3t}\\ e_{3,t-1}\\ e_{4t}\\ e_{4,t-1}\end{bmatrix}$$

(2) Transition equation:

$$\begin{bmatrix}\Delta c_t\\ \Delta c_{t-1}\\ \Delta c_{t-2}\\ \Delta c_{t-3}\\ e_{1t}\\ e_{1,t-1}\\ e_{2t}\\ e_{2,t-1}\\ e_{3t}\\ e_{3,t-1}\\ e_{4t}\\ e_{4,t-1}\end{bmatrix} = \begin{bmatrix}\phi(L)\mu_{s_t}\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}\phi_1 & \phi_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \psi_{11} & \psi_{12} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \psi_{21} & \psi_{22} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \psi_{31} & \psi_{32} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \psi_{41} & \psi_{42}\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix} \times \begin{bmatrix}\Delta c_{t-1}\\ \Delta c_{t-2}\\ \Delta c_{t-3}\\ \Delta c_{t-4}\\ e_{1,t-1}\\ e_{1,t-2}\\ e_{2,t-1}\\ e_{2,t-2}\\ e_{3,t-1}\\ e_{3,t-2}\\ e_{4,t-1}\\ e_{4,t-2}\end{bmatrix} + \begin{bmatrix}v_t\\ 0\\ 0\\ 0\\ \epsilon_{1t}\\ 0\\ \epsilon_{2t}\\ 0\\ \epsilon_{3t}\\ 0\\ \epsilon_{4t}\\ 0\end{bmatrix}$$

where $\phi(L)\mu_{s_t} = \mu_{s_t} - \phi_1\mu_{s_{t-1}} - \phi_2\mu_{s_{t-2}}$.

B.2 State–Space Representation 2

(1) Measurement equation:

$$\begin{bmatrix}\Delta y^*_{1t}\\ \Delta y^*_{2t}\\ \Delta y^*_{3t}\\ \Delta y^*_{4t}\end{bmatrix} = \begin{bmatrix}\lambda_1 & -\lambda_1\psi_{11} & -\lambda_1\psi_{12} & 0 & 0 & 0\\ \lambda_2 & -\lambda_2\psi_{21} & -\lambda_2\psi_{22} & 0 & 0 & 0\\ \lambda_3 & -\lambda_3\psi_{31} & -\lambda_3\psi_{32} & 0 & 0 & 0\\ \lambda_{40} & \lambda^*_{41} & \lambda^*_{42} & \lambda^*_{43} & \lambda^*_{44} & \lambda^*_{45}\end{bmatrix} \times \begin{bmatrix}\Delta c_t\\ \Delta c_{t-1}\\ \Delta c_{t-2}\\ \Delta c_{t-3}\\ \Delta c_{t-4}\\ \Delta c_{t-5}\end{bmatrix} + \begin{bmatrix}\epsilon_{1t}\\ \epsilon_{2t}\\ \epsilon_{3t}\\ \epsilon_{4t}\end{bmatrix}$$

where $y^*_{it} = y_{it} - \psi_{i,1}y_{i,t-1} - \psi_{i,2}y_{i,t-2}$, $i = 1, 2, 3, 4$; $\lambda^*_{41} = -\lambda_{40}\psi_{41} + \lambda_{41}$, $\lambda^*_{42} = -\lambda_{40}\psi_{42} - \lambda_{41}\psi_{41} + \lambda_{42}$, $\lambda^*_{43} = -\lambda_{41}\psi_{42} - \lambda_{42}\psi_{41} + \lambda_{43}$, $\lambda^*_{44} = \lambda_{42}\psi_{42} - \lambda_{43}\psi_{41}$, and $\lambda^*_{45} = -\lambda_{43}\psi_{42}$.

(2) Transition equation:

$$\begin{bmatrix}\Delta c_t\\ \Delta c_{t-1}\\ \Delta c_{t-2}\\ \Delta c_{t-3}\\ \Delta c_{t-4}\\ \Delta c_{t-5}\end{bmatrix} = \begin{bmatrix}\phi(L)\mu_{s_t}\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}\phi_1 & \phi_2 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix} \times \begin{bmatrix}\Delta c_{t-1}\\ \Delta c_{t-2}\\ \Delta c_{t-3}\\ \Delta c_{t-4}\\ \Delta c_{t-5}\\ \Delta c_{t-6}\end{bmatrix} + \begin{bmatrix}v_t\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix}$$

where $\phi(L)\mu_{s_t} = \mu_{s_t} - \phi_1\mu_{s_{t-1}} - \phi_2\mu_{s_{t-2}}$.

APPENDIX C

Gibbs Sampling for Tests of Duration Dependence

In this section we explain how $\gamma^* = [\gamma_0 \;\; \gamma_1]'$, $\gamma_{2,1}$, $\gamma_{3,1}$, d_2, and d_3 can be generated, given $\tilde{S}_T$ and thus $\tilde{S}^*_T$, $\tilde{N}_{0,-1}$, and $\tilde{N}_{1,-1}$. Of particular interest in this appendix are the posterior probabilities of $d_2 = 0$ and $d_3 = 0$, that is, the posterior probabilities of no duration dependence. The approach in this appendix is a straightforward application of Geweke (1994).

C.1 Generating γ_0, γ_1, $\gamma_{2,1}$, and $\gamma_{3,1}$

First, conditional on γ_{2,d_2} and γ_{3,d_3}, we generate $\gamma^* = [\gamma_0 \quad \gamma_1]'$. Define $\tilde{Y}_1$ and $\tilde{X}_1$ as the matrices of the left-hand-side and the right-hand-side variables in a regression, $S_t^* - \gamma_{2,d_2}(1 - S_{t-1})N_{0,t-1} - \gamma_{3,d_3}S_{t-1}N_{1,t-1} = \gamma_0 + \gamma_1 S_{t-1} + u_t$, $t = 2, 3, \ldots, T$. Given the prior distribution $\gamma^* \sim N(\underline{\gamma}^*, \underline{\omega}^{*2})$, the posterior distribution of γ^* is given by the following multivariate normal distribution:

$$\gamma^* \sim N(\bar{\gamma}^*, \bar{\omega}^*) \tag{C.1}$$

where $\bar{\omega}^* = (\underline{\omega}^{*-2} + \tilde{X}_1'\tilde{X}_1)^{-1}$ and $\bar{\gamma}^* = \bar{\omega}^*(\underline{\omega}^{*-2}\underline{\gamma}^* + \tilde{X}_1'\tilde{Y}_1)$.

Second, conditional on $\tilde{\gamma}^* = [\gamma_0 \quad \gamma_1]'$, γ_{3,d_3}, and on $d_2 = 1$, we generate $\gamma_{2,1}$. Define $\tilde{Y}_2$ and $\tilde{X}_2$ as the matrices of the left-hand-side and the right-hand-side variables in a regression $S_t^* - \gamma_0 - \gamma_1 S_{t-1} - \gamma_{3,d_3}S_{t-1}N_{1,t-1} = \gamma_{2,1}(1 - S_{t-1})N_{0,t-1} + u_t$, $t = 2, 3, \ldots, T$. Assuming the prior is given by a truncated normal distribution, $\gamma_{2,1} \sim N(\underline{\gamma}_2, \underline{\omega}_2^2)_{I[\gamma_2>0]}$, the posterior distribution of $\gamma_{2,1}$ is denoted by the following truncated normal distribution:

$$\gamma_{2,1} \sim N(\bar{\gamma}_2, \bar{\omega}_2)_{I[\gamma_{2,1}>0]} \tag{C.2}$$

where the subscript $I[\cdot]$ refers to an indicator function, $\bar{\omega}_2 = (\underline{\omega}_2^{-2} + \tilde{X}_2'\tilde{X}_2)^{-1}$, and $\bar{\gamma}_2 = \bar{\omega}_2(\underline{\omega}_2^{-2}\underline{\gamma}_2 + \tilde{X}_2'\tilde{Y}_2)$.

Finally, conditional on $\tilde{\gamma}^* = [\gamma_0 \quad \gamma_1]'$, γ_{2,d_2}, and on $d_3 = 1$, we generate $\gamma_{3,1}$. Define $\tilde{Y}_3$ and $\tilde{X}_3$ as the matrices of the left-hand-side and the right-hand-side variables in a regression, $S_t^* - \gamma_0 - \gamma_1 S_{t-1} - \gamma_{2,d_3}(1 - S_{t-1})N_{0,t-1} = \gamma_{3,1}S_{t-1}N_{0,t-1} + u_t$, $t = 2, 3, \ldots, T$. Assuming the prior is given by a truncated normal distribution, $\gamma_{3,1} \sim N(\underline{\gamma}_3, \underline{\omega}_3^2)_{I[\gamma_3<0]}$, the posterior distribution of $\gamma_{3,1}$ is denoted by the following truncated normal distribution:

$$\gamma_{3,1} \sim N(\bar{\gamma}_3, \bar{\omega}_3)_{I[\gamma_{3,1}<0]} \tag{C.3}$$

where the subscript $I[\cdot]$ refers to an indicator function, $\bar{\omega}_3 = (\underline{\omega}_3^{-2} + \tilde{X}_3'\tilde{X}_3)^{-1}$, and $\bar{\gamma}_3 = \bar{\omega}_3(\underline{\omega}_3^{-2}\underline{\gamma}_3 + \tilde{X}_3'\tilde{Y}_3)$.

C.2 Generating d$_2$ *and* d$_3$

As in George and McCulloch (1993) and Geweke (1994), d_2 and d_3 are generated componentwise by sampling consecutively from the conditional distribution

$$f(d_i | d_{j\neq i}, \tilde{\gamma}, \tilde{S}_T^*, \tilde{S}_T, \tilde{N}_{0,-1}, \tilde{N}_{1,-1}), \qquad i = 2, 3. \tag{C.4}$$

We follow the approach in Geweke (1994) in generating d_2 and d_3.

For example, d_2 can be generated from the conditional distribution

$$f(d_2 | \gamma^*, \gamma_{3,d_3}, \tilde{S}_T^*, \tilde{S}_T, \tilde{N}_{0,-1}, \tilde{N}_{1,-1}) \tag{C.5}$$

where distribution (C.5) is Bernoulli with probability

$$\Pr[d_2 = 1 | \gamma^*, \gamma_{3,d_3}, \tilde{S}_T^*, \tilde{S}_T, \tilde{N}_{0,-1}, \tilde{N}_{1,-1}] = \frac{p_2}{p_2 + (1 - p_2)\dfrac{a_1}{a_0}}. \tag{C.6}$$

Here p_2 is the prior probability, $\Pr[d_2 = 0]$, of no duration dependence for recessions and

$$a_1 = f(\gamma_{2,d_2} | \gamma^*, \gamma_{3,d_3}, \tilde{S}_T^*, \tilde{S}_T, \tilde{N}_{0,-1}, \tilde{N}_{1,-1}, d_2 = 1)_{I[\gamma_{2,1}>0]} \tag{C.7}$$

$$a_0 = f(\gamma_{2,d_2} | \gamma^*, \gamma_{3,d_3}, \tilde{S}_T^*, \tilde{S}_T, \tilde{N}_{0,-1}, \tilde{N}_{1,-1}, d_2 = 0). \tag{C.8}$$

The term a_1/a_0 in the right-hand side of equation (C.6) is the conditional Bayes factor in favor of $d_2 = 1$ ($\gamma_{2,d_2} > 0$) versus $d_2 = 0$ ($\gamma_{2,d_2} = 0$). This conditional Bayes factor can be shown as[15]

$$\frac{a_1}{a_0} = 2\exp\left(\frac{\bar{\gamma}_2}{2\bar{\omega}_2} - \frac{\underline{\gamma}_2}{2\underline{\omega}_2^2}\right)\left(\frac{\sqrt{\bar{\omega}_2}}{\underline{\omega}_2}\right)\left[1 - \Phi\left(\frac{\bar{\gamma}_2}{\sqrt{\bar{\omega}_2}}\right)\right] \tag{C.9}$$

where $\Phi(\cdot)$ is the c.d.f of the standard normal distribution. Thus based on a comparison of a drawing from the uniform distribution on [0, 1] with the probability calculated from equation (C.6), d_2 is generated as 0 or 1.

In a similar way, d_3 can be generated using the following posterior probability:

$$\Pr[d_3 = 1 | \gamma^*, \gamma_{2,d_2}, \tilde{S}_T^*, \tilde{S}_T, \tilde{N}_{0,-1}, \tilde{N}_{1,-1}] = \frac{p_3}{p_3 + (1 - p_3)\dfrac{b_1}{b_0}} \tag{C.10}$$

where b_0 and b_1 are appropriately defined as in equations (C.7) and (C.8); p_3 is the prior probability, $\Pr[d_3 = 0]$, of no duration dependence for booms; and the conditional Bayes factor in favor of $d_3 = 1$ ($\gamma_{3,d_3} < 0$) versus $d_3 = 0$ ($\gamma_{3,d_3} = 0$) is given by

$$\frac{b_1}{b_0} = 2\exp\left(\frac{\bar{\gamma}_3}{2\bar{\omega}_3} - \frac{\underline{\gamma}_3}{2\underline{\omega}_3^2}\right)\left(\frac{\sqrt{\bar{\omega}_3}}{\underline{\omega}_3}\right)\left[\Phi\left(-\frac{\bar{\gamma}_3}{\sqrt{\bar{\omega}_3}}\right)\right]. \tag{C.11}$$

[15] For a proof, refer to Geweke (1994).